EXERCISE: THE MITTAG–LEFFLER PROBLEM

Let X be a Riemann surface. The Mittag–Leffler problem is as follows: given the following data: a discrete set of points $\{z_\alpha\}$ in X, together with principal parts $f_\alpha \in \mathcal{M}_{z_\alpha}/\mathcal{O}_{z_\alpha} \cong \mathbb{C}[z_\alpha^{-1}]$, does there exist a meromorphic function $f \in \mathcal{M}(X)$ with only poles at the points z_α with principal parts f_α?

Exercise 1. As before, X is a Riemann surface.

a) Show that the assignment

$$\mathcal{P}(U) := \{(z_\alpha, f_\alpha),\ \{z_\alpha\} \subset U \text{ discrete},\ f_\alpha \in \mathcal{M}_{z_\alpha}/\mathcal{O}_{z_\alpha}\}$$

defines a sheaf on X which fits into an exact sequence

$$0 \longrightarrow \mathcal{O} \longrightarrow \mathcal{M} \longrightarrow \mathcal{P} \longrightarrow 0.$$

b) Show that data as in the Mittag–Leffler problem above determines a class $\{(z_\alpha, f_\alpha)\} \in H^0(X, \mathcal{P})$. Use the long exact sequence in cohomology to show that the problem has a positive solution if and only if $\delta(\{(z_\alpha, f_\alpha)\}) = 0 \in H^1(X, \mathcal{O})$. Because of this property, the class $\delta(\{(z_\alpha, f_\alpha)\})$ is called an *obstruction class*.

c) Given data as above, choose a covering $\{U_i\}_{i\in I}$ of X such that each U_i contains at most one of the points z_α, and let $f_i \in \mathcal{M}(U_i)$ be local solutions to the problem. Let $\{\chi_i\}_{i\in I}$ be a partition of unity subordinate to $\{U_i\}_{i\in I}$, equal to 1 in a neighborhood of $z_\alpha \in U_i$. Show that

$$\omega := \sum_i d''(\chi_i f_i),$$

initially defined on $X \backslash \{z_\alpha\}$, defines 1-form in $\mathcal{E}^{0,1}(X)$. Show the Mittag–Leffler problem has a positive solution if and only if the Dolbeault obstruction class $[\omega] = 0 \in \mathcal{E}^{0,1}(X)/d''\mathcal{E}(X)$

d) Show that under the Dolbeault isomorphism $H^1(X, \mathcal{O}) \cong \mathcal{E}^{0,1}(X)/d''\mathcal{E}(X)$, the obstruction class of $b)$ is mapped to that of $c)$